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THE DIRECTIONAL THEORY OF SCREWS.

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The term *screw* as used in this paper will be applied to the equivalent, in the language of Grassmann's system, of Ball's screw, as used in his "Theory of Screws."*

I will begin by giving Ball's own definition, and will then show how the meaning of the term is modified, and, as it appears to me, clarified, as it will be here employed.

On page 3 of the above mentioned book the following definition is given: A screw is "a straight line in space with which a definite linear magnitude, termed the pitch, is associated." *How* this linear magnitude is associated with the line does not at once appear, and it seems to me to be a matter of some difficulty to attach a clear geometric conception to the definition.

I shall define a screw to be *the sum of a point-vector and a plane-vector perpendicular to it*, the former being a *directed and posited line*, the latter the *product of two vectors*, hence a *directed*, but *not posited*, *plane*. To express this in symbols let S be a screw, e a point, ε a vector, a a scalar quantity (the pitch), and η a plane-vector perpendicular to ε and having an area numerically equal to the length of ε; i. e. $\eta = |\,\varepsilon$; then we shall write

$$S = e\varepsilon + a\eta = e\varepsilon + a\,|\,\varepsilon. \tag{1}$$

If $T\varepsilon = T\eta = 1$, we shall call S a *unit* screw. Fig. 1 will give a definite idea of a screw in our sense of the word. $e\varepsilon$ is the point-vector, fixed in length, direction, and position; $\eta (= |\,\varepsilon)$ a vector area, or plane-vector perpendicular to ε; and $a\eta$ the larger vector area, a multiple of the first.

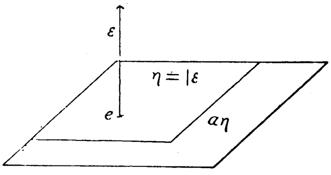

Fig. 1.

In mechanics the screw becomes a force and couple, the plane of the latter being perpendicular to the former, i. e. what Ball calls a *wrench*; or else a rotation and translation, the direction of the latter being parallel to the axis of the former, i. e. according to Ball's nomenclature, a *twist.*

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*The Theory of Screws: A Study in the Dynamics of a Rigid Body. By Robert Stawell Ball, LL.D., F. R. S. Dublin: Hodges, Foster, & Co.

A screw evidently belongs thoroughly to the realm of the Directional Calculus, and will not be easily or naturally treated by Cartesian methods; and yet Ball's treatment is, throughout, essentially Cartesian in its nature; i. e. he works always by *co-ordinates.* He also obtains purely geometric results mainly by mechanical processes and reasoning. That only Cartesian methods are employed appears at once from the statement immediately following the definition of a screw quoted above; viz:—

"If ∴ a screw be denoted by α, then α is not an algebraic quantity, and cannot occur in an algebraic equation." This simply means that a screw is not a *scalar* quantity, and cannot appear as a term in a *scalar* equation.

Returning to the value of the screw

$$S = e\varepsilon + a|\varepsilon,$$

we have, as particular cases,

$$S = e\varepsilon = \text{point-vector, when } a = 0,$$

and $S = a|\varepsilon =$ plane-vector, when $T\varepsilon = 0$ and $a = \infty$, with $aT\varepsilon$ a finite quantity.

The sum of any number of lines (point-vectors) is a screw, for

$$Se\varepsilon \equiv e_0\, \Sigma\varepsilon - e_0\, \Sigma\varepsilon + \Sigma e\varepsilon \equiv e_0\, \Sigma\varepsilon + \Sigma(e - e_0)\,\varepsilon. \tag{2}$$

The first term of the last member is a point-vector, and the second term a plane-vector not, in general, perpendicular to the first. Write $\Sigma\varepsilon = \varepsilon'$, $\Sigma(e - e_0)\varepsilon = \eta$, so that

$$\Sigma e\varepsilon \equiv e_0\varepsilon' + \eta; \tag{3}$$

then, if η is not perpendicular to ε', we can make it so by changing the position of the point-vector part as follows:

$$\Sigma e\varepsilon \equiv e_0\varepsilon' + e_0'\varepsilon' - e_0'\varepsilon' + \eta \equiv e_0'\varepsilon' + (e_0 - e_0')\,\varepsilon' + \eta \equiv e_0'\varepsilon' + \alpha\varepsilon' + \eta, \tag{4}$$

if $\alpha = e_0 - e_0'$. The vector α must be determined so that $\alpha\varepsilon' + \eta$ shall be perpendicular to ε'; that is, we must have $\varepsilon'\,|(\alpha\varepsilon' + \eta) = 0$. This gives α as extending from e_0 to a definite straight line parallel to ε', which is the locus of e_0'. If we take α perpendicular to ε', that is $\alpha|\varepsilon' = 0$, we find the perpendicular distance from e_0 to this line, for the condition above becomes $|\alpha . \varepsilon'^2 + \varepsilon'\,|\,\eta = 0$, or

$$\alpha = e_0 - e_0' = \frac{\eta\,|\,\varepsilon'}{\varepsilon'^2}. \tag{5}$$

This determines the *central axis* of the system of lines, and shows that $\Sigma e\varepsilon$ is identically equal to a certain definite screw.

To find the sum of a number of screws; i.e. *to determine S from the equation*

$$S = x_1 s_1 + x_2 s_2 + \ldots = \Sigma xs. \tag{6}$$

Let $s_1 = e_1 \varepsilon_1 + a_1 | \varepsilon_1$, with similar values for s_2, s_3, etc.; also write $S = p\sigma + z | \sigma$, in which p is a point, σ a vector, and z a scalar quantity, all of which must be determined in order to find S. Substituting these values in (6), we have

$$p\sigma + z | \sigma = \Sigma x e \varepsilon + \Sigma a x | \varepsilon,$$

or

$$e_0 \sigma + (p - e_0)\sigma + z | \sigma = e_0 \Sigma x \varepsilon + \Sigma [x(e - e_0)\varepsilon] + \Sigma a x | \varepsilon.$$

Write, for brevity, ε_1' for $e_1 - e_0$, ε_2' for $e_2 - e_0$, etc., and ρ for $p - e_0$, and equate separately the point-vector and plane-vector parts; thus we have

$$\sigma = \Sigma x \varepsilon, \tag{7}$$

and

$$\rho\sigma + z | \sigma = \Sigma x \varepsilon' \varepsilon + \Sigma a x | \varepsilon. \tag{8}$$

Equation (7) gives the value of σ. To find z, multiply equation (8) by σ, therefore

$$z\sigma^2 = \sigma \Sigma x \varepsilon' \varepsilon + \sigma \Sigma a x | \varepsilon. \tag{9}$$

Finally, multiply (8) into $| \sigma$, and we have

$$\rho\sigma | \sigma = (\Sigma x \varepsilon' \varepsilon + \Sigma a x | \varepsilon) | \sigma, \tag{10}$$

an equation which determines $p (= e_0 + \rho)$ as lying on a certain straight line parallel to σ. $\rho\sigma | \sigma \div \sigma^2$ is the projection of ρ on a plane perpendicular to σ; hence

$$\frac{1}{\sigma^2} (\Sigma x \varepsilon' \varepsilon + \Sigma a x | \varepsilon) | \sigma$$

is the perpendicular distance from e_0 to the axis of the resultant screw.

Product of two Screws.—Let $S_1 = e_1 \varepsilon_1 + a_1 | \varepsilon_1$ and $S_2 = e_2 \varepsilon_2 + a_2 | \varepsilon_2$ be two screws; then

$$S_1 S_2 = e_1 \varepsilon_1 e_2 \varepsilon_2 + a_2 . e_1 \varepsilon_1 | \varepsilon_2 + a_1 . e_2 \varepsilon_2 | \varepsilon_1 + a_1 a_2 . | \varepsilon_1 . | \varepsilon_2.$$

Now, remembering that $| \varepsilon_1$ and $| \varepsilon_2$ are lines (point-vectors) at ∞, we see that each of these terms is the product of two lines in solid space, and therefore scalar; the last term, being the product of two lines in the plane at ∞, is zero; and finally that $e_1 \varepsilon_1 | \varepsilon_2 = \varepsilon_1 | \varepsilon_2 = \varepsilon_2 | \varepsilon_1 = e_2 \varepsilon_2 | \varepsilon_1$. Hence

$$S_1 S_2 = e_1 \varepsilon_1 e_2 \varepsilon_2 + (a_1 + a_2) \varepsilon_1 | \varepsilon_2 = (a_1 + a_2) \cos \theta - \partial \sin \theta, \tag{11}$$

if ∂ is the perpendicular distance between $e_1 \varepsilon_1$ and $e_2 \varepsilon_2$, and θ is the angle between them. This expression Ball calls the "virtual coefficient" of two screws, and obtains it by a comparatively complicated mechanical process. The third member of (11) is obtained on the supposition that $T\varepsilon_1 = T\varepsilon_2 = 1$, which we shall always assume to be the case when the contrary is not expressly stated.

If $S_2 = S_1$, we have

$$S_1^2 = 2a_1. \tag{12}$$

From this equation we find at once the condition that a screw shall reduce to a line, viz:—

$$S^2 = 2a = 0. \tag{13}$$

When $S_1 S_2 = 0$, S_1 and S_2 are said to be *reciprocal* screws.

If p be some point, then pS is a plane, for it is the same as $pe\varepsilon + ap\eta$, i. e. the sum of two planes, and therefore a plane through their common line, which is the perpendicular from p on $e\varepsilon$.

The following formulæ are easily proved by putting in the value of S, S being taken as unit screw:—

$$S.p_1S \equiv ap_1, \tag{14}$$

$$Sp_1.Sp_2 \equiv Sp_1p_2.S - ap_1p_2, \tag{15}$$

$$Sp_1.Sp_2.Sp_3 \equiv a(p_1.p_2p_3S + p_2.p_3p_1S + p_3.p_1p_2S). \tag{16}$$

If we substitute P, a plane, for p, a point, we obtain a set of reciprocal formulæ, which may also be easily proved independently.

$$S.P_1S \equiv aP_1, \tag{17}$$

$$SP_1.SP_2 \equiv SP_1P_2.S - aP_1P_2, \tag{18}$$

$$SP_1.SP_2.SP_3 \equiv a(P_1.P_2P_3S + P_2.P_3P_1S + P_3.P_1P_2S). \tag{19}$$

Taking next a reference tetrahedron $e_0e_1e_2e_3$, let the six edges be represented by $\lambda_1, \ldots \lambda_6$, as shown in Fig. 2, i. e. $\lambda_1 = e_0e_1$, etc. Then it may be easily shown that any screw may be expressed in terms of these six edges with proper coefficients; thus,

$$S = \sum_1^6 x\lambda. \tag{20}$$

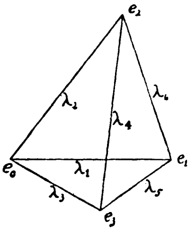

Fig. 2.

The x's may be subjected to one condition which affects only the *magnitude* of S, and not its *position* in any way; namely,

$$\Sigma x = C, \tag{21}$$

in which C may have any numerical value, such as unity. There remain, therefore, five independent and arbitrary constants for determining S, so that it may be subjected to five conditions.

Let $S_1 = \Sigma A\lambda$, $S_2 = \Sigma B\lambda$, . . . $S_5 = \Sigma E\lambda$, be any five screws; then the five conditions before mentioned may be taken as

$$\begin{aligned} SS_1 &= 0 = A_4x_1 + A_5x_2 + A_6x_3 + A_1x_4 + A_2x_5 + A_3x_6, \\ SS_2 &= 0 = B_4x_1 + B_5x_2 + \ldots, \\ & \ldots\ldots\ldots\ldots \\ SS_5 &= 0 = E_4x_1 + \ldots; \end{aligned} \tag{21a}$$

from which five equations the ratios x_2/x_1, x_3/x_1, etc., can be found, and thus the position of S completely determined. *Hence a screw may be found reciprocal to any five screws whatever.*

Any six screws may be taken as a reference system instead of λ_1, . . . λ_6; for, writing

$$S = x_1s_1 + x_2s_2 + \ldots + x_6s_6, \tag{22}$$

we have five arbitrary constants. If s_1' be reciprocal to $s_2, s_3, \ldots s_6$; s_2' to $s_1, s_3, \ldots s_6$, etc.; then multiplying (22) successively into s_1', s_2', etc., we have $s_1s_1'x_1 = Ss_1'$, $s_2s_2'x_2 = Ss_2'$, etc.; whence

$$S = \frac{Ss_1'}{s_1s_1'}.s_1 + \frac{Ss_2'}{s_2s_2'}.s_2 + \ldots + \frac{Ss_6'}{s_6s_6'}.s_6. \tag{23}$$

If $s_1, \ldots s_6$ be so taken that each is reciprocal to all the others, as may evidently be done, then these screws form a *co-reciprocal* reference system, and s_1 may be substituted for s_1', etc.; thus (23) becomes, if k_1, k_2, etc. be the pitches of s_1, s_2, etc.,

$$S = \tfrac{1}{2}\left(\frac{s_1}{k_1}.Ss_1 + \frac{s_2}{k_2}.Ss_2 + \ldots + \frac{s_6}{k_6}.Ss_6\right). \tag{24}$$

If $s_1, \ldots s_6$ be co-reciprocal, we obtain the pitch of S at once in simple form by squaring (22) or (24); thus,

$$\begin{aligned} S^2 = 2a &= 2(k_1x_1^2 + k_2x_2^2 + \ldots + k_6x_6^2) \\ &= \tfrac{1}{2}\left(\frac{(Ss_1)^2}{k_1} + \ldots + \frac{(Ss_6)^2}{k_6}\right). \end{aligned} \tag{25}$$

Reciprocal Screw Systems.—Let a variable screw S be subject to *one condition;* then it belongs to a screw system, or complex, of the fifth order. Let the condition be

$$Ss_1 = 0, \tag{26}$$

and let $s_2, \ldots s_6$ be five screws, each reciprocal to s_1; write also

$$S' = \sum_{1}^{6} x s, \tag{27}$$

so that S' is a variable screw dependent on $s_2, \ldots s_6$. Then we shall have

$$S's_1 = 0, \tag{28}$$

because $s_1s_2 = s_1s_3 = \ldots = 0$. Hence S' is subject to the same condition as S, and therefore they belong to the same system, of the fifth order because dependent on five screws. Equation (28) shows that s_1 is reciprocal to every screw of the system dependent on $s_2, \ldots s_6$.

Next let S be subject to *two conditions;* viz:—

$$Ss_1 = Ss_2 = 0; \tag{29}$$

then it belongs to a screw system, or complex, of the fourth order, because dependent on four screws. Let $s_3, \ldots s_6$ be four screws each reciprocal to both s_1 and s_2, and write

$$S' = \sum_3^6 xs. \tag{30}$$

Then $S's_1 = S's_2 = 0$, and S' belongs to the same system as S because subject to the same conditions. Write also

$$S'' = x_1s_1 + x_2s_2; \tag{31}$$

then

$$SS'' = 0, \tag{32}$$

so that every screw dependent on $s_3, \ldots s_6$ is reciprocal to every screw dependent on s_1 and s_2. These are *reciprocal screw systems.*

Reasoning in the same way we may show that a system subject to three conditions,

$$Ss_1 = Ss_2 = Ss_3 = 0, \tag{33}$$

is of the third order, being dependent on three screws, s_4, s_5, s_6, each of which is reciprocal to s_1, s_2, and s_3; while the system dependent on s_1, s_2, and s_3 is reciprocal to the first system.

In discussing these systems in detail it will be convenient to consider together the pairs of reciprocal systems. Conditional equations of higher degree than the first may, of course, be used, but such will not be treated in this paper.

Systems of the first and fifth orders.—The system of the first order is subject to five conditions, which, as we have previously seen, completely fix a single screw; hence this system consists of only one screw, and we will proceed to the consideration of its reciprocal, or the one-conditioned system of the fifth order, whose equation is

$$Ss_1 = 0, \tag{34}$$

Write $S \equiv p\sigma + z \,|\, \sigma$ and $s_1 = e_1\epsilon_1 + a_1 \,|\, \epsilon_1$; then

$$\begin{aligned} Ss_1 &= p\sigma e_1\epsilon_1 + (a_1 + z)\sigma \,|\, \epsilon_1 = p\sigma e_1\epsilon_1 + (a_1 + z)p\sigma \,|\, \epsilon_1 \\ &= p\sigma[e_1\epsilon_1 + (a_1 + z) \,|\, \epsilon_1] = Ls_1', \end{aligned}$$

say, in which $L = p\sigma$ and $s_1' = e_1\epsilon_1 + (a_1 + z) \,|\, \epsilon_1$. Hence we may write for equation (34)

$$Ls_1' = 0; \tag{35}$$

an equation of identical meaning, but arranged so as to have a variable *line* reciprocal to a screw whose *pitch* is variable. The advantage of this arrangement is, that we can first regard the pitch of s_1' as *constant*, determine the meaning of the equation in this case, and then generalize by making the pitch variable. First, then, we will suppose z to have some constant value, say zero, so that $s_1' = s_1$. The locus of L in this case is Plücker's "Linear Complex" as treated, entirely by Cartesian methods, in his *Neue Geometrie des Raumes*. Those who have read that work will notice the simplicity and brevity of the present methods as compared with his.

Let p be a variable point, and p_1 a fixed one, and write $L \equiv pp_1$, so that L varies in such a way as to pass always through a fixed point. Then (35) becomes

$$pp_1s_1 = 0. \tag{36}$$

This is the equation of a fixed plane p_1s_1, and shows that *all lines of the system* (35), *z being constant, which pass through a fixed point, lie in a fixed plane.*

Let P be a variable and P_1 a fixed plane, and write $PP_1 \equiv L$; then we have (35) in the form

$$PP_1s_1 = 0, \tag{37}$$

the plane equation of a fixed point, which shows that *all lines of the system that lie in a fixed plane pass through a fixed point of that plane*, a statement reciprocal to the preceding. Thus to every point in space corresponds a definite plane passing through it, and to every plane a definite point lying in it.

Now suppose z to vary; then the plane $p_1e_1\epsilon_1$ remains constant, while the other part of p_1s_1', viz: $(a + z)p_1 \,|\, \epsilon_1$, varies in magnitude; hence, as shown in Fig. 3, the plane p_1s_1' will revolve around the perpendicular from p_1 on $e_1\epsilon_1$ as an axis, through 180°, by giving all possible values to z. Hence it appears that every line in space that passes through a fixed point is the axis of one screw, of some definite pitch, of the system $Ss_1 = 0$; and, since this point may have any position whatever, it follows that *every* line in space is the axis of a single screw of this system. The lines cutting $e_1\epsilon_1$ at right angles

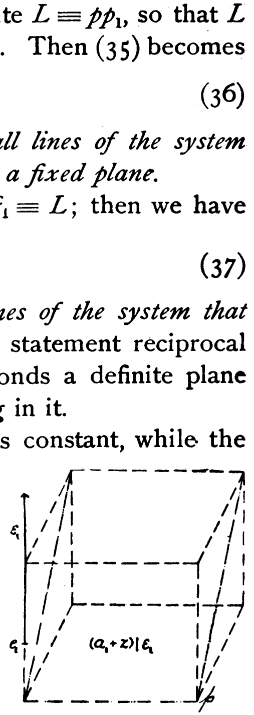

Fig. 3.

are peculiar lines, for each of them is the axis of an infinite number of screws belonging to the system. The line $e_1\epsilon_1$, the axis of s_1, is called the *axis of the system.*

Regarding z as zero again, we proceed to show that *to every line, L_1, in space there corresponds a line $s_1 . L_1 s_1 - a_1 L_1 = \varphi L_1$, say, such that the corresponding plane of any point on one line passes through the other, and, reciprocally, the corresponding point of any plane through one line lies on the other.* These lines are called *conjugate polars,* and it follows, from the relation between them just given, that every line that cuts a pair of conjugate polars belongs to the system $Ls_1 = 0$.

If L_1 belong to the system $Ls_1 = 0$, then $L_1 s_1 = 0$, and the conjugate polars are *coincident.* Hence every line of the system may be regarded as two coincident conjugate polars.

Let P_1 be any plane through φL_1 and $P_1 s_1$ its corresponding point; then we must have

$$P_1 \varphi L_1 = 0 = P_1 s_1 . L_1 s_1 - a_1 P_1 L_1,$$

whence

$$P_1 s_1 = \frac{a_1}{L_1 s_1} . P_1 L_1 ;$$

that is, the corresponding point is where P_1 cuts L_1. If we substitute p_1, a point, for P_1, we have

$$p_1 s_1 = \frac{a_1}{L_1 s_1} . p_1 L_1 ;$$

that is, the plane corresponding to p_1, a point on φL_1, passes through L_1.

To bring out still more clearly the reciprocal nature of L_1 and φL_1 we will show that $\varphi(\varphi L_1) = \varphi^2 L_1$ is a constant times L_1. We have

$$\begin{aligned}\varphi(\varphi L_1) &\equiv s_1 . s_1 (s_1 . s_1 L_1 - a_1 L_1) - a_1 (s_1 . s_1 L_1 - a_1 L_1) \\ &\equiv 2a_1 . s_1 . s_1 L_1 - 2a_1 . s_1 . s_1 L_1 + a_1^2 L_1 \\ &\equiv a_1^2 L_1. \quad \text{Q. E. D.}\end{aligned}$$

Next suppose L_1 to be at ∞, and replace it by η. Then to points on $\varphi\eta$ there correspond planes through these points and the line η at ∞; that is, a *system of parallel planes.* $\varphi\eta$ is always parallel to $e_1\epsilon_1$, the axis of s_1, for

$$\varphi\eta = s_1 . s_1 \eta - a_1 \eta = e_1 \epsilon_1 . s_1 \eta + a_1 (| \epsilon_1 . s_1 \eta - \eta);$$

that is, $\varphi\eta$ is the line $e_1\epsilon_1$ + a plane-vector not perpendicular to it, which is equivalent to a screw (in this case a *line*) parallel to $e_1\epsilon_1$. The system of parallel lines $\varphi\eta$, obtained by giving all values to η, is a system of *diameters* of the complex $Ss_1 = 0$. When $\eta = | \epsilon_1$, $\varphi\eta$ becomes simply $e_1\epsilon_1$; so that the axis of the complex is perpendicular to its corresponding planes.

Conversely, the conjugate polar of any diameter is a line at ∞. For, let $e\varepsilon_1$ be some diameter; then

$$\varphi e\varepsilon_1 \equiv s_1 . s_1 e\varepsilon_1 - a_1 e\varepsilon_1 \equiv (e_1\varepsilon_1 + a_1 | \varepsilon_1) a_1 - a_1 e\varepsilon_1 \equiv a_1 (e_1 - e) \varepsilon_1 + a_1^2 | \varepsilon,$$

a plane-vector, or line at ∞.

The three lines $e_1\varepsilon_1$, L_1, *and* φL_1 *have a common perpendicular, i. e. all three cut at right angles some fourth line.* For let L' be a line cutting at right angles $e_1\varepsilon_1$ and L_1, then it must satisfy the equations of condition $L'e_1\varepsilon_1 = 0$, $L'L_1 = 0$, and $L's_1 = 0$, the last by equation (11); hence $L'\varphi L_1 = 0$, so that L' *intersects* φL_1. Again we have $\varphi L_1 \equiv s_1 . s_1 L_1 - a_1 L_1$, which is of the form of equation (6), and equation (7) shows that, in this case, the directions of φL_1, ε_1, and L_1 are all parallel to one plane, and therefore perpendicular to L'.

The equation

$$\sum_1^6 (xS) = 0 \tag{38}$$

is the condition that six screws S_1, . . . S_6 *shall belong to one screw complex of the fifth order.* For, find a screw, S', reciprocal to each of the screws S_1, . . . S_5; then we have $S'S_1 = S'S_2 = S'S_3 = S'S_4 = S'S_5 = 0$, and consequently, also, $S'S_6 = 0$, so that all the screws belong to the system whose equation is $SS' = 0$. This condition may also be expressed in terms of S_1, . . . S_6 without scalar coefficients; for, multiply (38) successively by S_1, S_2, . . . S_6, obtaining thus six scalar equations, from which, eliminating x_1, . . . x_6, we have the following determinant, in which for convenience we have written 11 for S_1S_1, 12 for S_1S_2, etc.; viz:—

$$\begin{vmatrix} 11 & 12 & 13 & 14 & 15 & 16 \\ 21 & 22 & 23 & 24 & 25 & 26 \\ 31 & 32 & 33 & 34 & 35 & 36 \\ 41 & 42 & 43 & 44 & 45 & 46 \\ 51 & 52 & 53 & 54 & 55 & 56 \\ 61 & 62 & 63 & 64 & 65 & 66 \end{vmatrix} = 0. \tag{39}$$

Screw complexes of the second and fourth orders.—The equation of the second order complex may be written

$$S = x_1 s_1 + x_2 s_2. \tag{40}$$

Taking this as a particular case of equation (6), we have

$$\begin{aligned} \sigma &= x_1\varepsilon_1 + x_2\varepsilon_2, \\ \sigma^{\underline{2}} &= x_1^2 + x_2^2 + 2x_1x_2\varepsilon_1 | \varepsilon_2, \end{aligned} \tag{41}$$

$$\begin{aligned} -\rho\sigma|\sigma &\equiv \rho.\sigma^{\underline{2}} - \sigma.\rho|\sigma \\ &= \iota_3[x_1^2(c+\partial) + cx_2^2 + (2c+\partial)x_1x_2.\varepsilon_1|\varepsilon_2 + (a_2-a_1)x_1x_2T\varepsilon_1\varepsilon_2]; \end{aligned} \tag{42}$$

in which e_1 and e_2 have been taken at the ends of the common perpendicular between $e_1\varepsilon_1$ and $e_2\varepsilon_2$, e_0 is at some point on this perpendicular at a distance c from e_2, $e_1 - e_2 = \partial\iota_3$; whence

$$e_2 - e_0 = c\iota_3 = \varepsilon_2', \quad e_1 - e_0 = (c+\partial)\iota_3 = \varepsilon_1', \quad \iota_3|\varepsilon_1 = \iota_3|\varepsilon_2 = 0, \quad \text{and } \sigma|\iota_3 = 0.$$

If, in (42), we put σ for $\sigma.\rho|\sigma/\sigma^{\underline{2}}$ in the second term of the second member, ρ will extend from e_0 to the end of σ; if, then, we give x_1 and x_2 all values, $T\sigma$ will vary from 0 to ∞, while $U\sigma$ revolves about e_1e_2, so that the end of ρ moves on some surface, the locus of the axis of S in equation (40), which becomes

$$\rho = \sigma + \frac{\iota_3}{\sigma^{\underline{2}}}\{x_1^2(c+\partial) + cx_2^2 + [(2c+\partial)\varepsilon_1|\varepsilon_2 + (a_2-a_1)T\varepsilon_1\varepsilon_2]x_1x_2\}. \tag{43}$$

In this equation let $c = \partial = 0 = \varepsilon_1|\varepsilon_2$, that is, $e_1\varepsilon_1$ and $e_2\varepsilon$ intersect at right angles and e_0, e_1, and e_2 coincide; then (43) reduces to

$$\begin{aligned} \rho &= \sigma + \frac{\iota_3}{\sigma^{\underline{2}}}(a_2-a_1)x_1x_2 \\ &= x_1\iota_1 + x_2\iota_2 + \frac{(a_2-a_1)x_1x_2\iota_3}{x_1^2+x_2^2}, \end{aligned} \tag{44}$$

in which ι_1, ι_2 have been substituted for ε_1, ε_2. This is the equation of a third degree surface which Ball has called the *cylindroid*. We shall show that, by changing the origin and reference vectors, without making the conditions $c = \partial = \varepsilon_1|\varepsilon_2 = 0$, (43) can always be reduced to the form of (44). We will first intersect the surface by the plane through the origin perpendicular to ι_3, whose equation is $\rho = x_1\varepsilon_1 + x_2\varepsilon_2$. This value of ρ in (43) gives the coefficient of ι_3 as equal to zero, whence we find

$$\frac{x_2}{x_1} = \frac{A \pm \sqrt{(A^2+B)}}{2c}, \tag{45}$$

$$\begin{aligned} &\text{in which} \qquad & A &= -(2c+\partial)\varepsilon_1|\varepsilon_2 - (a_2-a_1)T\varepsilon_1\varepsilon_2, \\ &\text{and} & B &= -4c(c+\partial). \end{aligned} \tag{46}$$

As we have two definite values of x_2/x_1 the section must be two right lines, and we will see if it is possible to find such a value of c; that is, such a position of the origin on e_1e_2 as will make these two lines to be at right angles. Putting in the values of x_1 and x_2, we have

$$x_1\varepsilon_1 + x_2\varepsilon_2 = 2c\varepsilon_1 + [A + \sqrt{(A^2+B)}]\varepsilon_2, \quad \text{or } = 2c\varepsilon_1 + [A - \sqrt{(A^2+B)}]\varepsilon_2.$$

If these two vectors are perpendicular to each other, we must have

$$\{2c\epsilon_1 + [A + \surd(A^2 + B)]\epsilon_2\} \mid \{2c\epsilon_1 + [A - \surd(A^2 + B)]\epsilon_2\} = 0;$$

$$\therefore 4c^2 - B + 4cA\epsilon_1 \mid \epsilon_2 = 0 = 8c^2 + 4c\partial + 4c\epsilon_1 \mid \epsilon_2 \, [(a_1 - a_2)T\epsilon_1\epsilon_2 - (2c + \partial)\epsilon_1 \mid \epsilon_2];$$

$$\therefore 2c + \partial = \frac{(a_2 - a_1)\,\epsilon_1 \mid \epsilon_2}{T\epsilon_1\epsilon_2} = (a_2 - a_1) \cot \omega;$$

and $$c = \tfrac{1}{2}[(a_2 - a_1) \cot \omega - \partial], \tag{47}$$

if ω is the angle between ϵ_1 and ϵ_2. When c has this value the section is two perpendicular lines. For brevity write $a_2 - a_1 = d_a$; then (45) and (43) become respectively

$$\frac{x_2}{x_1} = \frac{-d_a \operatorname{cosec} \omega \pm \surd(d_a^2 + \partial^2)}{d_a \cot \omega - \partial}, \tag{48}$$

$$\rho = \sigma + \frac{\iota_3}{\sigma^2}\left[\tfrac{1}{2}(d_a \cot \omega + \partial)x_1^2 + \tfrac{1}{2}(d_a \cot \omega - \partial)x_2^2 + d_a x_1 x_2 \operatorname{cosec} \omega\right]. \tag{49}$$

We will next express σ in terms of unit vectors along these two perpendicular lines. Write

$$\sigma = x_1\epsilon_1 + x_2\epsilon_2 = x_1'\iota_1 + x_2'\iota_2;$$

whence $$\sigma^2 = x_1'^2 + x_2'^2,$$

$$x_2\epsilon_1\epsilon_2 = x_1'\epsilon_1\iota_1 + x_2'\epsilon_1\iota_2,$$

and $$x_1\epsilon_1\epsilon_2 = x_1'\iota_1\epsilon_2 + x_2'\iota_2\epsilon_2.$$

These plane-vectors all have the same unit, viz: $|\iota_3$, which may therefore be divided out; then, if we call θ the angle between ι_1 and ϵ_1, we have

$$x_1 = [x_1' \sin(\theta + \omega) - x_2' \cos(\theta + \omega)] \operatorname{cosec} \omega$$

and $$x_2 = (-x_1' \sin \theta + x_2' \cos \theta) \operatorname{cosec} \omega.$$

If we substitute these values of x_1 and x_2 in equation (49), and equate to zero the coefficients of $x_1'^2$ and $x_2'^2$, that equation will be reduced to the form of (44). Writing for brevity

$$\tfrac{1}{2}(d_a \cot \omega + \partial) = C, \qquad \tfrac{1}{2}(d_a \cot \omega - \partial) = D,$$

the above process gives

$$C \sin^2(\theta + \omega) + D \sin^2 \theta - d_a \operatorname{cosec} \omega \sin \theta \sin(\omega + \theta) = 0,$$

$$C \cos^2(\theta + \omega) + D \cos^2 \theta - d_a \operatorname{cosec} \omega \cos \theta \cos(\omega + \theta) = 0.$$

These equations are not independent, for on adding them θ is eliminated, and the resulting expression in terms of the constants vanishes identically. Hence they are consistent, and either will give the value of θ. We obtain for θ the value

$$\tan 2\theta = \frac{\partial + (a_2 - a_1) \cot \omega}{\partial \cot \omega - a_2 + a_1}, \tag{50}$$

and equation (49) reduces (dropping primes) to

$$\rho = x_1\iota_1 + x_2\iota_2 + \frac{x_1x_2\sqrt{[(a_2 - a_1)^2 + \partial^2]}}{x_1^2 + x_2^2} . \iota_3. \tag{51}$$

We have thus shown that the locus of the axis of S in equation (40) is always a cylindroid, a third degree ruled surface all of whose generators are perpendicular to the common perpendicular of s_1 and s_2. This surface is very fully discussed in Ball's "Theory of Screws." If we square equation (40) we obtain the value of the pitch,

$$z = a_1x_1^2 + a_2x_2^2 + x_1x_2[(a_1 + a_2) \cos \omega - \partial \sin \omega]. \tag{52}$$

If $z = 0$, so that S becomes a *line*, we have a quadratic for determining x_2/x_1, showing that there are two screws of zero pitch on the cylindroid. Similarly there are two screws of the second order system having *any* given pitch, and only two.

We will now consider the fourth order complex given by the equations

$$Ss_1 = Ss_2 = 0, \tag{53}$$

or, proceeding as in equations (34) and (35), by the equivalent equations

$$Ls_1' = Ls_2' = 0. \tag{54}$$

As in treating the fifth order system, we will first regard the pitch of S as zero, so that $s_1' = s_1$ and $s_2' = s_2$; also, as in that case, let $L \equiv pp_1 \equiv PP_1$. Then we have for any point p_1 two corresponding planes p_1s_1 and p_1s_2 whose common line belongs to the screw system under consideration. Thus, with z constant, *one line of the system passes through each point of space.* Similarly for any plane P_1 we have two corresponding points, P_1s_1 and P_1s_2, through which a line of the system passes, so that *one line of the complex lies in each plane in space.* It is evident that the present system consists of all the lines common to the two fifth order systems $Ls_1 = 0$ and $Ls_2 = 0$. The fifth order complex

$$L(s_1 + ks_2) = 0 \tag{55}$$

also contains every line of the present system, for (55) is satisfied when the equations $Ls_1 = 0$ and $Ls_2 = 0$ are satisfied. If k vary, the equation

$$S' = s_1 + ks_2 \tag{56}$$

represents a cylindroid to every screw of which every screw of the system $Ls_1 = Ls_2 = 0$ is reciprocal. If S' be a screw of zero pitch, i. e. a *line*, then L must *intersect* it, and we have seen that there are *two* such, real or imaginary, on every cylindroid. If S' be a line, we must have

$$S'^2 = 2z' = 0 = 2a_1 + 2ks_1s_2 + 2k^2a_2,$$

whence
$$2a_2k = -s_1s_2 \pm \sqrt{[(s_1s_2)^2 - 4a_1a_2]}. \tag{57}$$

Thus, when the values of k are real, *every line of the system $Ls_1 = Ls_2 = 0$ cuts two fixed straight lines;* viz. the two screws of zero pitch belonging to the cylindroid $s_1 + ks_2$. These two lines are the *directrices* of the system. The common perpendicular of s_1 and s_2 may be called the *axis* of each of the reciprocal systems $s_1 + ks_2$ and $Ls_1 = Ls_2 = 0$.

The above result may also be obtained thus: If any four values of L be given, say L_1, L_2, L_3, L_4, all reciprocal to both s_1 and s_2, then these lines determine the system, and its equation may be written

$$L = x_1L_1 + x_2L_2 + x_3L_3 + x_4L_4. \tag{58}$$

Now two lines may in general be drawn cutting L_1, L_2, L_3, and L_4 by generating a skew quadric on three of them which will be pierced by the fourth in two points real, coincident, or imaginary through which two lines real, coincident, or imaginary can be drawn cutting L_1, . . . L_4. Call these two lines A_1 and A_2; then, by (58), every line of the system will cut A_1 and A_2, and the equations of the complex might be written

$$LA_1 = LA_2 = 0. \tag{59}$$

The axis will be the common perpendicular of A_1 and A_2.

It appears at once, from this point of view, that one line of the system passes through each point of space, and lies in each plane in space, as we saw before; for a line can be drawn through any point cutting two given right lines, and any plane will cut A_1 and A_2 in two points fixing a line of the system.

Resuming the value pp_1 of L, let p_1 move along a line L_1; then the corresponding line of p_1 (i. e. the line of the system through p_1) must cut both φ_1L_1 and φ_2L_1, so that to every line in space there corresponds a skew quadric, of which this line is one directrix. Since the corresponding lines of L_1 also cut A_1 and A_2, it appears that L_1, φ_1L_1, φ_2L_1, A_1, and A_2 are all generators of the same generation of a skew quadric.

The directrices A_1 and A_2 are conjugate polars with reference to each of the fifth order systems, $Ls_1 = 0$ and $Ls_2 = 0$.

If we call the two values of k in (57) k_1 and k_2, we have $k_1k_2 = a_1/a_2$, or

$$k_2 = \frac{a_1}{a_2k_1}. \tag{60}$$

We are to show that $\varphi_1 A_1 = \text{constant} \times A_2$ and $\varphi_2 A_1 = \text{constant} \times A_2$. We have, since $A_1 = s_1 + k_1 s_2$,

$$\varphi_1 A_1 = \varphi_1 (s_1 + k_1 s_2) = s_1 . s_1^2 - a_1 s_1 + k_1 (s_1 . s_1 s_2 - a_1 s_2)$$

$$= a_1 s_1 + k_1 s_1 . s_1 s_2 - k_1 a_1 s_2.$$

But $\qquad \frac{1}{2} A_1^2 = 0 = a_1 + k_1 s_1 s_2 + a_2 k_1^2;$

whence $\qquad k_1 s_1 s_2 = -(a_1 + a_2 k_1^2);$

$$\therefore \varphi_1 A_1 = a_1 s_1 - k_1 a_1 s_2 - s_1 (a_1 + a_2 k_1^2) = -a_2 k_1^2 (s_1 + \frac{a_1}{a_2 k_1} . s_2)$$

$$= -a_2 k_1^2 (s_1 + k_2 s_2) = -a_2 k_1^2 A_2. \quad \text{Q. E. D.}$$

Similarly we find

$$\varphi_2 A_1 = \varphi_2 (s_1 + k_1 s_2) = -a_2 (s_1 + \frac{a_1}{a_2 k_1} . s_2) = -a_2 A_2.$$

We may also easily show in the same way, that A_1 and A_2 are conjugate polars with reference to the system $L(s_1 + k s_2) = 0$. For, calling the function in this case φ', we find

$$\varphi' A_1 = -a_2 (k - k_1)^2 A_2. \tag{61}$$

When $k_1 = k_2$, A_1 and A_2 *coincide*, and we have seen, in discussing the fifth order system, that in this case $A_1 (= A_2)$ is a line of the system $L s_1 = 0$ and also of the system $L s_2 = 0$, and consequently of the system $L(s_1 + k s_2) = 0$, and of course of the system $L s_1 = L s_2 = 0$.

If A_1 and A_2 are coplanar, since one line of the system passes through each point of space, and all lines cutting A_1 and A_2 belong to the system, it appears that the complex consists of all lines through the common point of A_1 and A_2, together with all lines in their plane.

We will now remove the restriction on the pitch of the fixed screws, taking z as variable. The equations of the complex are

$$L[e_1 \varepsilon_1 + (a_1 + z) | \varepsilon_1] = 0 = L[e_2 \varepsilon_2 + (a_2 + z) | \varepsilon_2];$$

whence
$$z = \frac{L s_1}{L | \varepsilon_1} = \frac{L s_2}{L | \varepsilon_2}, \tag{62}$$

an equation of the second degree in L, representing a line system of the fifth order and second degree, the locus of axes of screws of the system $S s_1 = S s_2 = 0$. Write ρp_1 for L, ρ being a variable vector and p_1 a fixed point, and we have a homogeneous equation in ρ representing a *cone of the second order*, the locus of the axes of all screws of the system $S s_1 = S s_2 = 0$ which pass through

p_1. If we write PP_1 for L, we have a homogeneous equation of the second degree in P, showing that the axes of screws of the system which lie in P_1 *envelope a conic.*

Again, write $p\varepsilon$ for L in (62), in order to find the locus of screws of the system having a given direction; thus we find

$$p\varepsilon \mid \varepsilon_2 . (p\varepsilon e_1 \varepsilon_1 + a_1 p\varepsilon \mid \varepsilon_1) = p\varepsilon \mid \varepsilon_1 . (p\varepsilon e_2 \varepsilon_2 + a_2 p\varepsilon \mid \varepsilon_2).$$

But $p\varepsilon \mid \varepsilon_2 = \varepsilon \mid \varepsilon_2$, etc.; whence

$$p\varepsilon \left(\frac{e_1\varepsilon_1}{\varepsilon \mid \varepsilon_1} - \frac{e_2\varepsilon_2}{\varepsilon \mid \varepsilon_2} \right) = a_2 - a_1, \tag{63}$$

the equation of a plane parallel to ε. If ε have successively all directions parallel to P_1, then (63) will represent a series of planes each cutting P_1 in a line, the axis of the only screw of the system $Ss_1 = Ss_2 = 0$ that lies in P_1 and is parallel to ε. Hence it appears that the envelope of the screws in P_1 is such a conic as has only one tangent in a given direction; that is *a parabola.*

If pp_1 be put for θ, and p_1 be taken on the cylindroid $S' = s_1 + ks_2$, then the cone of screws having its vertex at p_1 reduces to two planes through p_1. For, let

$$S' \equiv L' + z' \mid (\varepsilon_1 + k\varepsilon_2) = s_1 + ks_2 \equiv e_1\varepsilon_1 + a_1 \mid \varepsilon_1 + (e_2\varepsilon_2 + a_2 \mid \varepsilon_2)k;$$

$$\therefore L' = e_1\varepsilon_1 + (a_1 - z') \mid \varepsilon_1 + k\left[e_2\varepsilon_2 + (a_2 - z') \mid \varepsilon_2\right] = s_1' + ks_2',$$

say. If p_1 be on the cylindroid, we must have for some value of k

$$0 = p_1 L' = p_1 (s_1' + ks_2') = p_1 s_1 - z' p_1 \mid \varepsilon_1 + kp_1 s_2 - kz' p_1 \mid \varepsilon_2.$$

Equation (62) becomes

$$pp_1 s_2 . pp_1 \mid \varepsilon_1 = pp_1 \mid \varepsilon_2 . pp_1 s_1;$$

or, substituting for $p_1 s_1$ from the preceding equation,

$$pp_1 s_2 . pp_1 \mid \varepsilon_1 = pp_1 \mid \varepsilon_2 . p(z' p_1 \mid \varepsilon_1 - kp_1 s_2 + kz' p_1 \mid \varepsilon_2),$$

or

$$pp_1 s_2' . pp_1 \mid (\varepsilon_1 + k\varepsilon_2) = 0, \tag{63a}$$

an equation representing two planes through p_1, of which one is the corresponding plane to p_1 in the system $Ss_2 = 0$, and the other is perpendicular to the screw of the cylindroid on which p_1 lies. Since

$$p_1 L' \equiv p_1 (s_1' + ks_2') = 0,$$

whence

$$p_1 s_1' = - kp_1 s_2',$$

it appears that, when p_1 is on the cylindroid, its corresponding planes with reference to $Ss_1 = 0$ and $Ss_2 = 0$ are *coincident.*

The screw system of the third order.—Let the equations of such a system be

$$Ss_1 = Ss_2 = Ss_3 = 0, \tag{64}$$

which may also be written, as before,

$$Ls_1' = Ls_2' = Ls_3' = 0, \tag{65}$$

in which $$s_1' = s_1 + z\,|\,\varepsilon_1, \text{ etc.}$$

For any one of the equations of (64) we may substitute

$$S(s_1 + ks_2 + k's_3) = 0, \tag{66}$$

and for any one of (65) we may substitute

$$L(s_1' + ks_2' + k's_3') = 0. \tag{67}$$

To the system represented by (64) belong all screws common simultaneously to the three fifth order systems $Ss_1 = 0$, $Ss_2 = 0$, and $Ss_3 = 0$, and also belonging at the same time to all the systems obtained by giving different values to k and k' in (66).

As in previous cases, we will first consider the case when $z = 0$, or $s_1' = s_1$, etc., so that (67) becomes

$$L(s_1 + ks_2 + k's_3) = 0, \tag{68}$$

and every line of the complex is reciprocal to every screw obtained by giving different values to k and k' in the expression

$$s_1 + ks_2 + k's_3.$$

All of these screws whose pitch is *zero* must be *cut* by L; let such a screw, or line, be A; then writing

$$A = s_1 + ks_2 + k's_3,$$

we have $$\tfrac{1}{2}A^2 = 0 = a_1 + k^2a_2 + k'^2a_3 + ks_1s_2 + kk's_2s_3 + k's_3s_1. \tag{69}$$

This equation is satisfied by an infinite number of values of k and k', and hence A may occupy an infinite number of positions. In every one of these positions it is cut by every L of equation (68); hence the system L and the system A must be the generators of the two generations of a skew quadric. This result may also be obtained as follows: Put pp_1 for L; then p_1s_1, p_1s_2, p_1s_3 are the three corresponding planes of p_1. All lines of the system $Ls_1 = 0$ passing through p_1 lie in p_1s_1, and similarly for the other two. If these three planes have a common line, this line must belong to the system $Ls_1 = Ls_2 = Ls_3 = 0$. The condition that they shall have a common line is

$$p_1s_1 \cdot p_1s_2 \cdot p_1s_3 = 0.$$

If p_1 move subject to this condition it must always lie on some line of the system, and hence the locus of p_1 will be the same as that of L. The product of three planes in general is their common point times a scalar quantity, and, as p_1 is evidently a common point in this case, the product may be written

$$p_1 \cdot (s_2 \cdot s_3 p_1 \cdot p_1 s_1).$$

Now p_1, being a unit point, cannot be zero; hence the scalar part must be zero, and we have, dropping the suffix of the p_1, because the point is to be a variable,

$$- s_2 . s_3 p . p s_1 \equiv p s_1 . s_2 . s_3 p = 0. \qquad (70)$$

This, being an equation of the second degree in p, represents a quadric, which from the nature of the case must be, in general, an hyperboloid of one sheet.

To treat the case when the pitch of S varies, we have only to put $s_1' = s_1 + z \mid \varepsilon_1$ for s_1, etc. in equation (70), thus obtaining

$$p s_1' . s_2' . s_3' p \equiv p(s_1 + z \mid \varepsilon_1) . (s_2 + z \mid \varepsilon_2) . (s_3 + z \mid \varepsilon_3) p = 0. \qquad (71)$$

For any given value of z this equation represents a skew quadric. Thus the locus of L in (65) is an infinity of skew quadrics obtained by giving all values to z in (71). If two reciprocal screws have respectively the pitches z and $-z$, by equation (11) they must *intersect;* thus all screws of the system $Ss_1 = Ss_2 = Ss_3 = 0$ having the pitch z intersect all screws of the system $S' = x_1 s_1 + x_2 s_2 + x_3 s_3$ having the pitch $-z$, and *vice versa.* Thus, for any given numerical value of z, the screws of the two reciprocal complexes are the two systems of generators of the skew quadric (71) with that value of z.

Suppose (71) to pass through some definite point p_1; then the equation must be satisfied when we put p_1 for p, and we have

$$p_1 (s_1 + z \mid \varepsilon_1) . (s_2 + z \mid \varepsilon_2) . (s_3 + z \mid \varepsilon_3) p_1 = 0, \qquad (72)$$

a cubic in z, from which three values of z may be determined such as will satisfy the equation. Hence three surfaces of the system (71) pass through p_1, and consequently three screws of the system $Ss_1 = Ss_2 = Ss_3 = 0$, as well as three of the system $S' = x_1 s_1 + x_2 s_2 + x_3 s_3$, pass through any given point.

Let z_1, z_2, z_3 be the three roots of (72); then, as this equation was found by making the condition that the corresponding planes of p_1 should have a common line, it follows that, with the above values of z, the products of these planes two by two will be the lines of the system (65) which pass through p_1. The three lines are therefore

$$\begin{aligned} &p_1 (s_1 + z_1 \mid \varepsilon_1) . p_1 (s_2 + z_1 \mid \varepsilon_2), \\ &p_1 (s_1 + z_2 \mid \varepsilon_1) . p_1 (s_2 + z_2 \mid \varepsilon_2), \\ &p_1 (s_1 + z_3 \mid \varepsilon_1) . p_1 (s_2 + z_3 \mid \varepsilon_2). \end{aligned} \qquad (73)$$

If p_1 be at ∞, call it ε, substitute in (72) and expand. It will be found that the z^3 term, and all terms containing z^2 disappear, leaving an equation of the first degree giving one finite value of z. Hence through every point at ∞, i. e. in any given direction, we have only *one* screw of finite pitch. The same would appear from equation (63), for this is the equation of a plane containing all screws parallel to ε, common to the systems $Ss_1 = 0$ and $Ss_2 = 0$. We shall have in the same way a plane containing all screws parallel to ε and common to the systems $Ss_2 = 0$ and $Ss_3 = 0$, and the common line of these two planes will belong to all three systems. This line is given then by any two of the equations

$$p\varepsilon\left(\frac{e_1\varepsilon_1}{\varepsilon\,|\,\varepsilon_1} - \frac{e_2\varepsilon_2}{\varepsilon\,|\,\varepsilon_2}\right) = a_2 - a_1,$$

$$p\varepsilon\left(\frac{e_2\varepsilon_2}{\varepsilon\,|\,\varepsilon_2} - \frac{e_3\varepsilon_3}{\varepsilon\,|\,\varepsilon_3}\right) = a_3 - a_2, \qquad (74)$$

$$p\varepsilon\left(\frac{e_3\varepsilon_3}{\varepsilon\,|\,\varepsilon_3} - \frac{e_1\varepsilon_1}{\varepsilon\,|\,\varepsilon_1}\right) = a_1 - a_3,$$

which evidently represent planes having a common line, since, on adding, they vanish identically.

We will conclude this paper with a few words as to polar and conjugate screws.

Let φS be any linear, self-conjugate screw function of S, i. e. each term is of the first degree in S, and not scalar, and

$$S\varphi S' = S'\varphi S;$$

and let S' be such a screw as will satisfy the condition

$$S\varphi S' = 0; \qquad (75)$$

then S and S' are *conjugate* screws, while S and φS, as well as S' and $\varphi S'$, are *polar* screws with reference to a screw system of the fifth order and second degree whose equation is

$$S\varphi S = 0. \qquad (76)$$

Taking the above as a definition of the terms conjugate and polar as applied to screws, we will show that the corresponding lines which we have called "conjugate polars" in discussing the fifth order complex, are rightly named, being both conjugate and polar to each other. We showed that when

$$\varphi L = s_1 . Ls_1 - a_1 L,$$

then $$\varphi^2 L = a_1^2 L.$$

Write $$\varphi L = L',$$

then we have $$L\varphi L' = L\varphi^2 L = a_1^2 L^2 = 0,$$

which proves that L and L' are both conjugate and polar to each other.

Whatever be the form of the linear self-conjugate function φ, *there may always be found one system of six screws which are all conjugate to each other and also co-reciprocal;* that is, if the screws be $s_1, s_2, \ldots s_6$, they will satisfy the conditions

$$s_1\varphi s_2 = s_1\varphi s_3 = \ldots = s_1\varphi s_6 = s_2\varphi s_3 = \ldots = s_2\varphi s_6 = \ldots = s_5\varphi s_6 = 0$$

and

$$s_1 s_2 = s_1 s_3 = \ldots = s_1 s_6 = s_2 s_3 = \ldots = s_2 s_6 = \ldots = s_5 s_6 = 0.$$

We have seen that five conditions are necessary and sufficient to determine one screw, and hence thirty conditions will determine six screws; but the two sets of conditions above are thirty in number, so that the six screws may always be found. To determine them, we have only to solve the equation

$$\varphi S = gS,$$

or $$(\varphi - g)S = 0; \qquad (77)$$

in which g is a scalar. For, let S_1, g_1 and S_2, g_2 be two pairs of values which satisfy (77); then $\varphi S_1 = g_1 S_1$ and $\varphi S_2 = g_2 S_2$; whence $S_1\varphi S_2 = g_2 S_1 S_2$, so that, if $S_1 S_2 = 0$, $S_1\varphi S_2 = 0$ also. Let $s_1, s_2, \ldots s_6$ be any six screws whatever, and multiply (77) by each of them; therefore $s_1(\varphi - g)S = S(\varphi - g)s_1 = 0$, and so on for the other five. If S is to be reciprocal simultaneously to each of these six screws $(\varphi - g)s_1$, $(\varphi - g)s_2$, etc., we must have by equation (38),

$$\sum_1^6 [x(\varphi - g)s] = 0. \qquad (78)$$

Eliminating $x_1, x_2, \ldots x_6$ as in equation (39) we obtain an equation of the sixth degree in g, from which six values of g may be found. If any one of these values be inserted in five of the equations $S(\varphi - g)s_1 = 0$, $S(\varphi - g)s_2 = 0$, etc., the corresponding value of S may be found as in equations (21*a*).

Whatever may have been the original form of φ, if it be expressed in terms of these six co-reciprocal, conjugate screws, it is easily shown that it will have the form

$$\varphi S = \Sigma \left(\frac{g_n}{2a_n} S_n . S_n S\right), \qquad (79)$$

in which n has the values 1 to 6 inclusive.